1 Cubic formula

Let

$$(x - r_1)(x - r_2)(x - r_3) = x^3 + ax^2 + bx + c.$$

Matching coefficients, we obtain

$$\begin{aligned} a &= -(r_1 + r_2 + r_3) \\ b &= r_1 r_2 + r_1 r_3 + r_2 r_3 \\ c &= -r_1 r_2 r_3 \end{aligned}$$

Let

$$\begin{aligned} y &= r_1 + \omega r_2 + \omega^2 r_3 \\ z &= r_1 + \omega^2 r_2 + \omega r_3 \end{aligned}$$

where $\omega = e^{2\pi i/3} = \frac{-1+i\sqrt{3}}{2}$. Then

$$\begin{aligned} y^3 + z^3 &= -(r_1 + r_2 - 2r_3)(r_2 + r_3 - 2r_1)(r_1 + r_3 - 2r_2) \\ &= 2r_1^3 + 2r_2^3 + 2r_3^3 - 3r_1^2 r_2 - 3r_2^2 r_3 - 3r_3^2 r_1 - 3r_1 r_2^2 - 3r_2 r_3^2 - 3r_3 r_1^2 + 12 r_1 r_2 r_3 \\ &= -2a^3 + 9ab - 27c \\ y^3 z^3 &= (r_1^2 + r_2^2 + r_3^2 - r_1 r_2 - r_2 r_3 - r_1 r_3)^3 = (-a^2 - 3b)^3 \end{aligned}$$

Hence

$$(x - y^3)(x - z^3) = x^2 - (y^3 + z^3)x + y^3 z^3 = x^2 + (2a^3 - 9ab + 27c)x + (-a^2 - 3b)^3.$$

Solving for y^3 and z^3 with the quadratic formula yields

$$\begin{aligned} y &= \left(\frac{-2a^3 + 9ab - 27c + \sqrt{(2a^3 - 9ab + 27c)^2 + 4(a^2 + 3b)^3}}{2} \right)^{1/3} \\ z &= \left(\frac{-2a^3 + 9ab - 27c - \sqrt{(2a^3 - 9ab + 27c)^2 + 4(a^2 + 3b)^3}}{2} \right)^{1/3} \end{aligned}$$

Now we solve the linear system

$$\begin{aligned} a &= -(r_1 + r_2 + r_3) \\ y &= r_1 + \omega r_2 + \omega^2 r_3 \\ z &= r_1 + \omega^2 r_2 + \omega r_3 \end{aligned}$$

and we get

$$\begin{aligned} r_1 &= \frac{1}{3}(-a + y + z) \\ r_2 &= \frac{1}{3}(-a + \omega^2 y + \omega z) \\ r_3 &= \frac{1}{3}(-a + \omega y + \omega^2 z) \end{aligned}$$

which expresses r_1, r_2, r_3 in terms of a, b, c by way of the previously obtained expressions for y and z, and completes the derivation of the cubic formula.

2 Quartic formula

Let $x^4 + ax^3 + bx^2 + cx + d$ be a polynomial with four roots r_1, r_2, r_3, r_4, so

$$(x - r_1)(x - r_2)(x - r_3)(x - r_4) = x^4 + ax^3 + bx^2 + cx + d.$$

Then

$$\begin{aligned} a &= -(r_1 + r_2 + r_3 + r_4) \\ b &= r_1r_2 + r_1r_3 + r_1r_4 + r_2r_3 + r_2r_4 + r_3r_4 \\ c &= -(r_1r_2r_3 + r_1r_2r_4 + r_1r_3r_4 + r_2r_3r_4) \\ d &= r_1r_2r_3r_4 \end{aligned}$$

Let

$$\begin{aligned} t_1 &= (r_1 + r_2)(r_3 + r_4) \\ t_2 &= (r_1 + r_3)(r_2 + r_4) \\ t_3 &= (r_1 + r_4)(r_2 + r_3) \end{aligned}$$

By direct calculation, we obtain

$$(x - t_1)(x - t_2)(x - t_3) = x^3 - 2bx^2 + (b^2 + ac - 4d)x + (c^2 + a^2d - abc).$$

Using the cubic formula, solve for t_1, t_2, and t_3 in terms of a, b, c, d. Form the quadratic polynomial

$$(x - (r_1 + r_2))(x - (r_3 + r_4)) = x^2 + ax + t_1$$

with roots

$$\begin{aligned} r_1 + r_2 &= \frac{-a + \sqrt{a^2 - 4t_1}}{2} \\ r_3 + r_4 &= \frac{-a - \sqrt{a^2 - 4t_1}}{2} \end{aligned}$$

Now, using the equation $(t_2 + t_3 - t_1)/2 = r_1r_2 + r_3r_4$, form the quadratic polynomial

$$(x - r_1r_2)(x - r_3r_4) = x^2 - \frac{(t_2 + t_3 - t_1)}{2}x + d$$

with roots

$$\begin{aligned} r_1r_2 &= \frac{t_2 + t_3 - t_1 + \sqrt{(t_2 + t_3 - t_1)^2 - 16d}}{4} \\ r_3r_4 &= \frac{t_2 + t_3 - t_1 - \sqrt{(t_2 + t_3 - t_1)^2 - 16d}}{4} \end{aligned}$$

Finally, using the known values of $r_1 + r_2$ and r_1r_2, solve the quadratic equation

$$\begin{aligned} (x - r_1)(x - r_2) &= x^2 - (r_1 + r_2)x + r_1r_2 \\ &= x^2 - \left(\frac{-a + \sqrt{a^2 - 4t_1}}{2}\right)x + \frac{t_2 + t_3 - t_1 + \sqrt{(t_2 + t_3 - t_1)^2 - 16d}}{4} \end{aligned}$$

for r_1 and r_2 to obtain the roots of the original quartic in terms of a, b, c, d.